A new look at the pseudo-incompressible solution to Lamb's problem of hydrostatic adjustment

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ABSTRACT

The pseudo-incompressible solution to Lamb's problem of the hydrostatic adjustment of the atmosphere is revisited using a generalized formulation which allows time variation of the base state. It is found that the generalized pseudo-incompressible equations do reproduce the compressible solution of the nonlinear hydrostatic adjustment in the limit of small heating.

1 Introduction

In a series of papers by Bannon (1995a,b; 1996) Lamb's problem of hydrostatic adjustment of the atmosphere is analyzed using the compressible equations and several different formulations of the anelastic approximation, including the pseudo-incompressible formulation (Durran 1989). The solution of the pseudo-incompressible equation with a time-independent base state is found to fundamentally differ from the solution of the compressible equations in both the linearized and nonlinear cases. In this note we demonstrate that if the base state pressure is allowed to vary with time, the correct equilibrium solution for a sufficiently small heating rate is, in fact, obtained from the appropriately generalized pseudo-incompressible equation.

2 Generalized Pseudo-Incompressible Equation

Here we derive the generalized pseudo-incompressible equation using low Mach number asymptotics. Rather than assume a time-independent base state *a priori*, we expand the compressible equations in terms of the Mach number, assumed to be small, in order to derive a set of approximate "soundproof" equations with as few assumptions as possible.
The compressible set of equations governing motion in a dry ideal gas atmosphere can be written in terms of the primitive variables $\mathbf{u}, P, \rho$, and T (velocity, pressure, density, and temperature, respectively):

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla P = -g\mathbf{k} \tag{2.1}$$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0 \tag{2.2}$$

$$P = \rho R T \tag{2.3}$$

$$\rho c_p \frac{DT}{Dt} = \frac{DP}{Dt} + H. \tag{2.4}$$

Here H is the net heating rate per unit volume. Consistent with the notation used by Durran (1989), we scale the dynamics variables as follows $(x,y) = L(x^*,y^*), z = Dz^*, (u,v) = U(u^*,v^*), w = Ww^*, t = \tau t^*, \rho = R_0\rho^*, P = R_0 c_s^2 P^*$, such that $U = L/\tau$, $W = D/\tau$, and c_s is the speed of sound. For generality, we assume that $D/L = W/U = \alpha \leq 1$, i.e. the time scale $\tau = L/U = D/W$ is the same for vertical and horizontal motions, but the horizontal and vertical velocity scales may not be of the same magnitude. We define the Mach number $M = U/c_s$; then $W/c_s = \alpha M$.
The nondimensionalized momentum equations can then be written

$$\frac{Du^*}{Dt^*} + (\frac{1}{M^2})\frac{1}{\rho^*}\frac{\partial P^*}{\partial x^*} = 0 \quad (2.5)$$

$$\frac{Dv^*}{Dt^*} + (\frac{1}{M^2})\frac{1}{\rho^*}\frac{\partial P^*}{\partial y^*} = 0 \quad (2.6)$$

$$\frac{Dw^*}{Dt^*} + (\frac{1}{\alpha^2 M^2})\frac{1}{\rho^*}\frac{\partial P^*}{\partial z^*} = -(\frac{1}{\alpha^2 M^2})\frac{gD}{c_s^2} \quad (2.7)$$

We now expand $\mathbf{u}^*, \rho^*$ and P^* in M, which we assume to be small:

$$\begin{aligned}
\mathbf{u}^*(\mathbf{x}^*,t^*) &= \mathbf{u_0^*}(\mathbf{x}^*,t^*) + M\mathbf{u_1^*}(\mathbf{x}^*,t^*) + M^2\mathbf{u_2^*}(\mathbf{x}^*,t^*) + O(M^3),\\
\rho^*(\mathbf{x}^*,t^*) &= \rho_0^*(\mathbf{x}^*,t^*) + M\rho_1^*(\mathbf{x}^*,t^*) + M^2\rho_2^*(\mathbf{x}^*,t^*) + O(M^3),\\
P^*(\mathbf{x}^*,t^*) &= P_0^*(\mathbf{x}^*,t^*) + MP_1^*(\mathbf{x}^*,t^*) + M^2P_2^*(\mathbf{x}^*,t^*) + O(M^3).
\end{aligned}$$

For the vertical length scale D such that $g^* = \frac{D}{c_s^2}g = O(1)$, gathering terms of $O(M^{-2}), O(M^{-1})$, and $O(M^0)$ yields

$$\begin{aligned}
O(M^{-2}) &: \frac{\partial P_0^*}{\partial x^*} = \frac{\partial P_0^*}{\partial y^*} = 0, \quad \frac{\partial P_0^*}{\partial z^*} = -\rho_0^* g^*\\
O(M^{-1}) &: \frac{\partial P_1^*}{\partial x^*} = \frac{\partial P_1^*}{\partial y^*} = 0, \quad \frac{\partial P_1^*}{\partial z^*} = -\rho_1^* g^*\\
O(M^0) &: \rho_0^*\frac{Du_0^*}{Dt^*} + \frac{\partial P_2^*}{\partial x^*} = \rho_0^*\frac{Dv_0^*}{Dt^*} + \frac{\partial P_2^*}{\partial y^*} = 0, \quad \rho_0^*\frac{Dw_0^*}{Dt^*} + \frac{\partial P_2^*}{\partial z^*} = -\rho_2^* g^*
\end{aligned}$$

We define $\bar{P}^* = P_0^* + MP_1^*$. From this analysis it follows that $\bar{P}^*$ is a function of z and t only; it follows from $\frac{\partial P_0^*}{\partial z^*} = -\rho_0^* g$ and $\frac{\partial P_1^*}{\partial z^*} = -\rho_1^* g$ that $\bar{\rho}^* = \rho_0^* + M\rho_1^*$ is a function of z and t only as well. Letting $\bar{P} = \rho_0 c_s^2 \bar{P}^*$ and $\bar{\rho} = \rho_0\bar{\rho}^*$, we define $\bar{T}(z,t) = \frac{\bar{P}}{R\bar{\rho}}$.
(As an aside, if $D \ll \frac{c_s^2}{g}$, then $g^* \ll 1$, and the above equations simplify to $\frac{\partial \bar{P}}{\partial z^*} = 0$, i.e. the vertical variation of the base state can be neglected.)
Defining the perturbational pressure $P' = P - \bar{P}$ and perturbational density, $\rho' = \rho - \bar{\rho}$, redimensionalizing the $O(M^0)$ terms in the momentum equations gives a familiar form:

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\bar{\rho}}\nabla P' = -\frac{\rho'}{\bar{\rho}}g\mathbf{k} \quad (2.8)$$

In the above derivation, there is a single length scale for each coordinate direction, which is the length scale associated with the predominant fluid motion. If one introduces a new horizontal length scale, $L_\Omega \gg L$, that of the much larger domain within which the motions take place, then the nondimensionalized horizontal momentum equations for the large-scale variation become

$$\frac{Du^*}{Dt^*} + (\frac{1}{M^2})(\frac{L}{L_\Omega})\frac{1}{\rho^*}\frac{\partial P^*}{\partial x^*} = 0 \quad (2.9)$$

$$\frac{Dv^*}{Dt^*} + (\frac{1}{M^2})(\frac{L}{L_\Omega})\frac{1}{\rho^*}\frac{\partial P^*}{\partial y^*} = 0 \quad (2.10)$$

If $\frac{L}{L_\Omega} = O(M^2)$, then the $O(M^0)$ equations in fact allow for long wavelength variation of the base state in the horizontal plane. We refer the reader to Durran (1998) for more details; this extension is not in fact necessary for the problem of hydrostatic adjustment.

In modeling the earth's atmosphere it is typical to neglect the time dependence of $\bar{P}$, since the phenomena of interest do not alter the base state. However, Lamb's problem is not a typical mesoscale phenomenon, and we will consider the more general case. It is important to note that the low Mach number assumption in itself does not force a time-independent base state. In fact, low Mach number asymptotics for combustion in a closed domain yield $\bar{P}(t)$; stellar convection has been modeled using a radially and temporally varying base state (see, e.g., Latour et al. 1976).

We form a pressure evolution equation by combining the continuity and energy equations:

$$\frac{1}{\gamma P}\frac{DP}{Dt} + (\nabla\cdot\mathbf{u}) = \frac{RH}{c_p P}, \tag{2.11}$$

where $\gamma = c_p/c_v$. As in Durran (1989) the pseudo-incompressible equation follows from this equation. We substitute $\bar{P}$ for P, and rewrite the above as

$$(\nabla\cdot\mathbf{u}) + \frac{1}{\gamma\bar{P}}\frac{D\bar{P}}{Dt} = \frac{RH}{c_p\bar{P}}, \tag{2.12}$$

If we now assume that $\frac{\partial\bar{P}}{\partial t} = 0$, then

$$\frac{1}{\gamma\bar{P}}\frac{D\bar{P}}{Dt} = \frac{w}{\gamma\bar{P}}\frac{\partial\bar{P}}{\partial z} = \frac{w}{\bar{\rho}\bar{\theta}}\frac{\partial(\bar{\rho}\bar{\theta})}{\partial z}$$

so

$$\bar{\rho}\bar{\theta}(\nabla\cdot\mathbf{u} + \frac{1}{\gamma\bar{P}}\frac{D\bar{P}}{Dt}) = \nabla\cdot(\bar{\rho}\bar{\theta}\mathbf{u}) = \frac{RH}{c_p}\frac{\bar{\rho}\bar{\theta}}{\bar{P}} = \frac{H}{c_p\bar{\pi}}, \tag{2.13}$$

where the base state Exner function, $\bar{\pi} = (\frac{\bar{P}}{P_{00}})^{R/c_p} = (\frac{R}{P_{00}}\bar{\rho}\bar{\theta})^{R/c_v}$. However, if we retain the time-dependence of $\bar{P}$, then

$$\frac{1}{\gamma\bar{P}}\frac{D\bar{P}}{Dt} = \frac{w}{\gamma\bar{P}}\frac{\partial\bar{P}}{\partial z} + \frac{1}{\gamma\bar{P}}\frac{\partial\bar{P}}{\partial t}$$

and the constraint becomes

$$\nabla\cdot(\bar{\rho}\bar{\theta}\mathbf{u}) = \frac{H}{c_p\bar{\pi}} + \frac{1}{R\gamma\bar{\pi}}\frac{\partial\bar{P}}{\partial t} \tag{2.14}$$

3 Nonlinear Hydrostatic Adjustment

We summarize the problem as presented by Bannon (1996a): at time $t = 0$, "the atmosphere is subjected to an instantaneous heating rate per unit volume of the form "

$$H = Q_{00}\, q(z)\, \delta(t)$$

"where Q_{00} is the constant amplitude of the energy per unit volume (positive for heat added; negative for heat extracted) and δ is the Dirac delta function. Here $q(z)$ describes the vertical distribution of the heating with the origin $z = 0$ taken to be the center of the heating. The atmosphere is semi-infinite with a flat rigid lower boundary at $z = -b$. " To derive the solution of the compressible equation set Bannon observes that instantaneous heating occurs at constant volume, and produces an immediate change in P and T (and hence in θ) but not ρ. The equilibration of the fluid then occurs at constant entropy, i.e. $D\theta/Dt = 0$. The solution is given in terms of the displacement ζ as a function of initial vertical location z_0 of a fluid parcel. The compressible solution to the nonlinear problem is

$$\zeta(z_0) = \int_{-b}^{z_0} \left(\left[1 + \frac{Q_{00}\, q(z')}{c_v \bar{\rho}(z') \bar{T}(z')} \right]^{\frac{1}{\gamma}} - 1 \right) dz'. \tag{3.1}$$

In the limit of small Q_{00}, the solution can be written

$$\zeta(z_0) = Q_{00} \int_{-b}^{z_0} \frac{q(z')}{c_p \bar{\rho}(z') \bar{T}(z')} dz' = \frac{R\, Q_{00}}{c_p} \int_{-b}^{z_0} \frac{q(z')}{\bar{P}(z')} dz'. \tag{3.2}$$

In this formulation $\bar{\rho}(z), \bar{T}(z)$, and $\bar{P}(z)$ define the time-independent base state for the compressible equations, and are identical to $\bar{\rho}(z,0), \bar{T}(z,0)$ and $\bar{P}(z,0)$ in the pseudo-incompressible equations. We note that this integral can be interpreted in either a Lagrangian or Eulerian frame, since the fluid is assumed not to move during the heating.

We also note that while the pressure does change during the heating and equilibration phases, the final equilibrium pressure of each parcel of air is identical to the initial pressure of that parcel, i.e.

$$\int_{0^-}^{\infty} \frac{DP}{Dt}\, dt = 0.$$

Fundamental to the pseudo-incompressible approximation (and other anelastic approximations) is that the time scale of acoustic equilibration is much smaller than the time scale of fluid motion, so we must restate Lamb's problem in this context. Here we posit that the heating happens on a time scale such that the pressure equilibrates over the time scale of the heating (equivalent to thinking of the sound waves as infinitely fast). Thus the heating occurs at constant *pressure* rather than constant *volume.* This has the effect of allowing displacement of the fluid particles during rather than after the heating. We must modify the heating profile accordingly, so that rather than defining $H = Q_{00}\, q(z)\, \delta(t)$, we set $H = Q_{00}\, q(z,t)\, \delta(t)$, such that $Dq/Dt = 0$.

In order to find the pseudo-incompressible solution, Bannon integrates the constraint in the form (2.13) under the assumption that $\bar{\rho}$ and $\bar{\theta}$ depend only on z. Thus

$$\frac{d}{dz}(\bar{\rho}(z)\bar{\theta}(z)\zeta) = \frac{\bar{\theta}(z)\; Q_{00}\; q(z)}{c_p \bar{T}(z)}$$

can be integrated to yield

$$\zeta(z_0) = \frac{Q_{00}}{c_p \bar{\rho}(z_0) \bar{\theta}(z_0)} \int_{-b}^{z_0} \frac{\bar{\theta}(z') q(z')}{\bar{T}(z')} dz'. \tag{3.3}$$

This solution does not reduce to the compressible solution even for small heating.

However, with the time-dependent base state, and the formulation of the problem such that $D\bar{P}/Dt = 0$ for each parcel of fluid (i.e. heating at constant pressure), the expression (2.12) for the divergence becomes,

$$\nabla \cdot \mathbf{u} = \frac{R\, Q_{00}\, q(z,t)}{c_p \bar{P}} - \frac{1}{\gamma \bar{P}} \frac{D\bar{P}}{Dt}$$

and can be integrated in space and time along each particle path (i.e. in a Lagrangian frame), using $Dq/Dt = 0$, to define

$$\zeta(z_0) = \frac{R\, Q_{00}}{c_p} \int_{-b}^{z_0} \frac{q(z_0')}{\bar{P}(z_0')} dz_0'. \tag{3.4}$$

We note that this is exactly the compressible solution, (3.2), in the limit of small Q_{00}.
In both the compressible solution and the pseudo-incompressible solution, the initial and final pressure of each parcel of air is the same. We have just demonstrated that the displacement fields are identical for small heating. Hydrostatic equilibrium and the equation of state then dictate that the density and temperature fields must also match.

4 Conclusions

In Bannon (1996a), several different formulations of the anelastic equations are evaluated, and none are found which reproduce the compressible solution to the nonlinear problem of hydrostatic adjustment for a time-independent base state. In the small amplitude limit, only the equations given by Dutton and Fichtl (1969) are found to accurately reproduce the thermodynamic state variables. However, they do not correctly predict the displacement of the fluid.
Here we have demonstrated that in the small amplitude limit, the pseudo-incompressible equation set does in fact reproduce the correct compressible solution, both the thermodynamic variables and the displacement, if the base state atmosphere is allowed to vary in time in response to the heating, and if the problem is correctly reformulated to make sense in the context of "soundproof" equations. Finally, we comment that it is in fact natural that the response to the heating should be absorbed by the base state rather than the perturbational fields; it is the base state which represents the hydrostatic balance, and the problem is exactly one of hydrostatic adjustment.

Acknowledgments. This work was supported, through the U. S. Department of Energy under Contract DE-AC03-76SF00098, by the Defense Special Weapons Agency IACRO98-3017 and the DOE Applied Mathematical Sciences Program.

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